Theorem 6.2.1 (Markov Inequality). Let $X$ be a random variable such that $P(X \geq 0) = 1$. Then for every real number $t > 0$,

$$P(X \geq t) \leq \frac{1}{t} E(X).$$

Observations about the theorem:
- The theorem applies to any nonnegative random variable (e.g., exponential, Bernoulli, binomial, Poisson, gamma and beta but not Gaussian).
- It applies to $X$ as long as $X$ is almost surely nonnegative.
- $X$ can be discrete, continuous or neither.
- It gives information about the probability of big values of $X$.
- It gives an upper bound on the probability of such values.
- It is used to prove the Chebyshev Inequality.
- There may be much better upper bounds on $P(X \geq t)$ than the one given in the theorem.

Proof of 6.2.1. Assume $X$ is continuous with density $f(x)$ (your text has a proof for the discrete case).

$$E(X) = \int_{-\infty}^{\infty} x f(x) \, dx$$
$$= \int_{X < t} x f(x) \, dx + \int_{X \geq t} x f(x) \, dx$$
$$\geq \int_{X \geq t} x f(x) \, dx$$
$$\geq t \int_{X \geq t} f(x) \, dx$$
$$= t P(X \geq t)$$
$$\frac{1}{t} E(X) \geq P(X \geq t) \square$$

Exercise. Justify each step in the proof above.

Example. Let $X$ be an exponential random variable with $\beta = 2$. Since $E(X) = 1/2$, the Markov Inequality asserts that $P(X \geq t) \leq \frac{1}{2t}$ for any $t > 0$.

The exact value of the probability is (show this!)

$$P(X \geq t) = \int_{t}^{\infty} 2e^{-2x} \, dx = e^{-2t}$$

![Graph showing Markov Inequality Bound and Exact Value](image-url)
Section 6.2: The Chebyshev Inequality

**Theorem 6.2.2 (Chebyshev Inequality).** Let $X$ be a random variable with $E(X) = \mu$ and $V(X) = \sigma^2$. Then for every real number $z > 0$,

$$P \left( \left| \frac{X - \mu}{\sigma} \right| \geq z \right) \leq \frac{1}{z^2}.$$ 

Observations about the theorem:

- The theorem applies to *any* $X$ for which $V(X)$ exists (since this implies that $E(X)$ exists).
- $X$ can be discrete, continuous or neither.
- The theorem gives information about the probability of big deviations of $X$ from its mean (measured in units of the standard deviation $\sigma$).
- It gives an upper bound on the probability of such values.
- There may be much better upper bounds on $P(|(X - \mu)/\sigma| \geq z)$ than the one given in the theorem.
- The *dimensionless* quantity $(X - \mu)/\sigma$ is the random variable $X$ in *standard units*.
- The theorem is used to prove the Law of Large Numbers.

**Proof of 6.2.1.** Let $Y = \left( \frac{X - \mu}{\sigma} \right)^2$. Note that:

$$E(Y) = E \left[ \left( \frac{X - \mu}{\sigma} \right)^2 \right]$$

$$= \frac{1}{\sigma^2} E \left[ (X - \mu)^2 \right]$$

$$= \frac{1}{\sigma^2} E \left( X^2 - 2 \mu X + \mu^2 \right)$$

$$= \frac{1}{\sigma^2} \left( E(X^2) - 2 \mu E(X) + \mu^2 \right)$$

$$= \frac{1}{\sigma^2} \left( \sigma^2 + \mu^2 - 2 \mu^2 + \mu^2 \right)$$

$$= 1$$

Since $Y \geq 0$, $P(Y \geq 0) = 1$. Hence, we can apply the Markov Inequality:

$$P \left( \left| \frac{X - \mu}{\sigma} \right| \geq z \right) = P \left( \left( \frac{X - \mu}{\sigma} \right)^2 \geq z^2 \right)$$

$$= P(Y \geq z^2)$$

$$\leq \frac{1}{z^2} E(Y)$$

$$= \frac{1}{z^2}$$

**Exercise.** Justify each step in the proof above.
Example. Let $X$ be an exponential random variable with $\beta = 2$. Since $E(X) = 1/2$ and $V(X) = 1/4$, the Chebyshev Inequality asserts that $P(|(X - 1/2)/(1/4)| \geq z) \leq 1/z^2$ for any $z > 0$. The exact value of the probability is (show this!)

$$P \left( \left| \frac{X - 1/2}{1/4} \right| \geq z \right) = \begin{cases} \exp \left[-\left(1 + \frac{1}{z^2}\right)\right] - \exp \left[-\left(1 - \frac{1}{z^2}\right)\right] + 1 & \text{if } z \leq 2 \\ \exp \left[-\left(1 + \frac{1}{z^2}\right)\right] & \text{if } z > 2. \end{cases}$$

Example. Let $X$ be a standard normal random variable (i.e., a Gaussian with $\mu = 0$ and $\sigma = 1$). We can compare the exact value of

$$P \left( \left| \frac{X - \mu}{\sigma} \right| \geq z \right)$$

to the upper bound provided by the Chebychev inequality.

<table>
<thead>
<tr>
<th>$z$</th>
<th>Exact</th>
<th>Chebychev</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.317</td>
<td>1.00</td>
</tr>
<tr>
<td>2</td>
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<td>0.25</td>
</tr>
<tr>
<td>3</td>
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<td>0.11</td>
</tr>
</tbody>
</table>
Worksheet: Markov and Chebychev Inequalities

1. Let \( X \) be a Bernoulli random variable with parameters \( n = 2 \) and \( \theta = 1/2 \).
   
   a) Make a plot comparing the exact value of \( P(X \geq t) \) to the Markov Inequality upper bound.

   b) Make a plot comparing the exact value of

   \[
P \left( \left| \frac{X - \mu}{\sigma} \right| \geq z \right)
   \]

   to the upper bound provided by the Chebyshev Inequality.